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Cluster percolation and critical behaviour in spin models and $SU(N)$ gauge theories

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Abstract

The critical behaviour of several spin models can be simply described as percolation of some suitably defined clusters or droplets: the onset of the geometrical transition coincides with the critical point, and the percolation exponents are equal to the thermal exponents. It is still unknown whether, given a model, one can define at all the droplets. In the cases where this is possible, the droplet definition depends in general on the specific model at study and can be quite involved. We propose here a simple general definition for the droplets: they are clusters obtained by joining nearest-neighbour spins of the same sign with some bond probability $p_{\rm CK}$, which is the minimal probability that still allows the existence of a percolating cluster at the critical temperature T_c. This definition has been recently introduced and indeed satisfies the conditions required for the droplets, for many classical spin models, discrete and continuous, in two dimensions. By means of lattice Monte Carlo simulations we show here that the definition works as well in three dimensions. In particular, our prescription allows us to describe exactly the confinement–deconfinement transition of $SU(N)$ gauge theories as Polyakov loop percolation.

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1. Introduction

Percolation theory [1] is the ideal framework for a geometrical description of phase transitions. The percolation phenomenon takes place when geometrical *clusters*, formed by elementary objects of some system, stick to each other giving rise to an infinite network that spans the whole system. Thermal phase transitions in spin models take place in an analogous way: the clusters are regions of correlated spins which represent local realizations of the new phase.

The temptation to identify a continuous thermal phase transition with a simple percolation transition is then very strong. This identification is possible provided one can establish a one-to-one correspondence between thermal and percolation variables. The main percolation variables are:

- the percolation strength P, i.e. the probability that a site chosen at random belongs to a percolating cluster;
- the average cluster size S,

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s} \tag{1}$$

where n_s is the number of clusters with s sites and the sums exclude eventual percolating clusters.

Suppose we have defined how to group the spins of the configurations of a given model in clusters. Such clusters are the 'physical' clusters or *droplets* [2] of the model if the following conditions are satisfied:

- the percolation point coincides with the thermal critical point;
- the connectedness length (average cluster radius) diverges as the thermal correlation length (same exponent);
- the percolation strength P near the threshold varies like the order parameter m of the model (same exponent);
- the average cluster size S diverges as the physical susceptibility χ (same exponent).

In the Ising model, if one introduces the temperature-dependent bond probability $p_B = 1 - \exp(-2J/kT)$ (J is the Ising coupling) and joins nearest-neighbouring spins of the same sign with this probability, the corresponding site-bond clusters are indeed the critical droplets of the model in any dimension [3, 4].

Nobody can yet say whether the phase transition of every model can be geometrically described as a percolation transition, i.e. whether one can always define the droplets. The Ising result, which is valid more in general for the q-state Potts model, can easily be extended to several spin systems, both discrete and continuous [5–7]. In general, one finds that each interaction between a pair of spins corresponds to a bond in the percolation picture with an analogous bond probability as in Ising. In models with several spin–spin interactions of the same type (all ferro- or anti-ferromagnetic), one can still define a percolation picture [6] by putting bonds between any pair of interacting spins with some probability, but the picture becomes quite involved: if two interacting spins are far from each other, the geometrical bond between them looks virtual, as the two spins are geometrically disconnected.

Recent investigations [8, 9] aimed at recovering the importance of the role of geometrical connectivity in the mapping between percolation and thermal critical behaviour. In [9] it was shown that, for a wide class of bidimensional models, one can define simple site-bond clusters which show all features the droplets should have. For the models where a rigorous mapping between percolation and critical behaviour is possible, such site-bond clusters are in general different from the 'exact' droplets (see [5–7]), which are in general more complex, even if their behaviour at criticality is identical. Moreover, the result remains valid as well for models with competitive interactions (e.g., ferromagnetic + antiferromagnetic), for which an exact definition of the droplets is, at present, missing. In [9] one examined theories with centre symmetry $Z(2)$ and $Z(3)$, such that their critical behaviour is in the universality class of the model obtained by removing all interactions except the nearest-neighbour one (Ising for $Z(2)$, 3-state Potts for $Z(3)$). Therefore, for these models the nearest-neighbour spin–spin coupling is the fundamental interaction which determines the behaviour at the phase transition. This is probably the reason why, if one just considers geometrical connections between nearest neighbours, weighted by some suitable bond probability, the corresponding clusters are at least a good approximation of the critical droplets of the model. The previous argument is of course independent of the number d of space dimensions of the system. For this reason we

believe that the result of [9] must be valid in general, i.e. for $d > 2$ as well. In this paper we provide numerical evidence, based on Monte Carlo simulations, that our droplet definition holds true also in three dimensions.

We stress that the original target of our investigations was to provide a geometrical description in terms of percolation of the confinement–deconfinement transition in $SU(N)$ gauge theories. Early attempts focused on $SU(2)$ pure gauge theory, whose deconfining transition is second order and in the universality class of the Ising model [10]. The strategy we followed at that stage was to approximate the gauge model by means of simpler Polyakov loop effective theories for which an exact droplet definition exists [11, 12]. In this way one finds only an approximate solution of the problem, which strongly depends on the specific lattice regularization one chooses. The studies reported in [11, 12], in fact, concerned lattice regularizations of $2 + 1$ and $3 + 1$ $SU(2)$ gauge theory which correspond to the so-called strong coupling limit: the number N_τ of lattice spacings in the temperature direction is then constrained to be small, e.g., $N_\tau = 2, 3, 4$. The proposed percolation picture fails for $N_\tau \gg 4$. This seemed to us unsatisfactory: the droplet prescription we propose here solves the problem in a simple and general way.

The paper is divided as follows: in section 2 we introduce our droplet definition; in section 3 we present the results of our simulations, distinguishing between spin models and $SU(2)$ pure gauge theory; and finally the conclusions of our work are presented.

2. The droplet definition

Our droplet candidates are clusters built by joining nearest-neighbour spins of the same sign with a bond probability p_B: they are then uniquely defined once we specify p_B. In [9], the following criterion was proposed: p_B must equal the minimal probability p_{CK} that still makes percolation possible at the critical temperature T_c (we call such probability p_{CK} after Coniglio and Klein [4]). This special minimal probability depends on the model under study. We recall that p_{CK} is in general a function of the temperature, as in the Ising model (where $p_{\mathrm{CK}}(T) = 1 - \exp(-2J/kT)$). We are only interested in the behaviour at the transition, i.e. near T_c. By approximating the bond weight $p_{\mathrm{CK}}(T)$ with its value $p_{\mathrm{CK}}(T_c)$, the power law behaviour of the percolation variables at criticality would not be appreciably affected and the critical exponents would remain the same. Besides, we calculated the exponents through finite size scaling fits of the percolation variables at T_c, so that the above-mentioned approximation does not affect at all our numerical results. This is why, referring to p_B (p_{CK}), we use the term 'value' instead of 'expression'.

We shall verify the validity of the 'criterion of the minimal probability' in three dimensions through computer simulations of two models: the $O(2)$ spin model and $SU(2)$ lattice gauge theory.

3. Results

3.1. Numerical analysis

Our aim is to investigate the percolation transition of special site-bond clusters, determining in particular the percolation temperature and the critical exponents.

To produce the equilibrium configurations we made use of standard Monte Carlo algorithms, such as Metropolis or heat bath; for some models we adopted cluster updates, such as the Wolff algorithm, which allows us to reduce sensibly the correlation of the data and save a lot of CPU time. At each iteration, once the configuration to be examined is

determined, all lattice sites are grouped in clusters by means of the algorithm devised by Hoshen and Kopelman [13]; for the cluster labelling we have always used free boundary conditions. After that we are left with a set of clusters of various sizes, and we can calculate the percolation variables. If a cluster connects the top with the bottom face of the lattice, we say that it percolates. We recall that below the upper critical dimension ($d = 4$ for Ising-like systems) and in the infinite volume limit, there can be just a single percolating cluster at the critical point (see [14]). Nevertheless, since we are forced to simulate our systems on finite lattices, it may happen that one finds more than one spanning cluster in the same configuration, although this is quite rare. In such cases we consider all spanning clusters as 'pieces' of one larger percolating cluster, whose size is then given by the sum of the individual sizes. Besides the percolation strength P and the average cluster size S, we also calculate the size S_M of the largest cluster of the configuration, since from it one can derive the fractal dimension D of the spanning cluster at the threshold, which is the exponent relating the size s and the radius R of a cluster, according to the formula $s \propto R^D$. At each iteration the energy density ϵ and the lattice average m of the order parameter of the thermal transition were also stored; this is necessary to determine the thermal critical temperature of some of the models we have studied, for which it is unknown.

The first step is of course the determination of the percolation temperature. This can be effectively done by using the percolation cumulant Π, whose definition and properties have been given in [8, 9]. We recall in particular that the value of Π at the percolation threshold labels a well-defined universality class.

In our numerical analysis we had no choice but tuning by hand the value of the bond probability p_B, starting from 1 downwards. We recall that p_B is the only free parameter we have, and that, once we fix it to some value, the percolation problem is uniquely and unambiguously determined. We always work at the critical temperature T_c of the model, and the question is to check whether, for the current value of p_B, the critical temperature T_c is also the percolation temperature T_p of the corresponding clusters. To see this we can simply check whether the percolation cumulant Π (at T_c) takes the same value for each of the lattices we used. We assumed that this is the case if values of Π for the three largest lattices (at least) agree within their error bars. If T_c turns out to be a percolation point, we determine the critical exponents by means of standard finite-size scaling techniques. For the finite-size scaling laws of the percolation variables at the threshold, we simply consider the leading behaviour

$$P(T_p) \propto L^{-\beta_p/\nu_p} \tag{2}$$

$$S(T_p) \propto L^{\gamma_p/\nu_p} \tag{3}$$

$$S_M(T_p) \propto L^D \tag{4}$$

where L is the lattice side, and β_p, γ_p are the exponents that rule the power law behaviour at criticality of P and S, respectively. The error we introduce by neglecting corrections to scaling is certainly smaller than the uncertainty on p_{CK}. In order to improve the precision of the fits and to keep disturbing finite-size effects under control, for each model four to six different lattices were used. However, we found very often that the slopes of our log–log plots did not change appreciably by omitting the smallest lattices, which shows that finite-size effects are not important. The errors of the critical exponents are the uncertainties of the slopes we obtained from the finite-size scaling log–log fits at T_c (except when $p_B = p_{\mathrm{CK}}$, see below).

It is quite a delicate matter to determine the error of the value of p_{CK}. We adopted the following numerical criterion: since we always sit at T_c, we check for which p_B values the cumulants Π for the three largest lattices coincide within error bars. We define p_{CK}

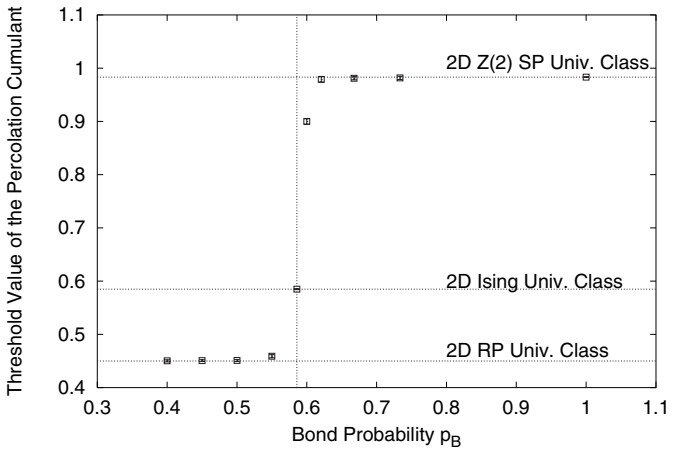

Figure 1. 2D Ising model: variation of Π at the percolation temperature T_p with the bond weight p_B.

as the minimal p_B for which the centre values of Π for the three largest lattices differ from each other by less than the smallest σ on the three numbers. We then decrease p_B until the Π values of all lattices at T_c are offset by more than one σ: this gives us the lower bound of p_{CK}. In three dimensions, the same criterion can be adopted to find the upper bound of p_{CK}. Here we would have to increase p_B starting from p_{CK}.

As far as the final errors on the critical indices of the droplets are concerned, we performed finite-size scaling fits at T_c by varying p_B within the range determined by the bounds of p_{CK}. For any fit we stored the best fit values for the exponents as well as their uncertainties. After scanning the whole range centred at p_{CK}, we determined the total variation of the fit parameters, which we took as total error. The centre values of the exponents are the best fit parameters obtained by doing the fit exactly at p_{CK}.

From a strictly numerical point of view one should take care to interpret the data of the simulations when p_B is close to p_{CK}. In this case, in fact, the system finds itself in the neighbourhood of a discontinuity and if the lattice is not large enough, its behaviour is influenced by that. The simulations on small lattices would produce configurations which represent a sort of mixture of the two situations at $p_B = p_{\mathrm{CK}}$ and $p_B \neq p_{\mathrm{CK}}$. To recover the real behaviour of the system, one should then go to very large lattices and disregard the small ones.

3.2. Spin models

In [9] we have seen that, in two dimensions, there is a whole range of p_B values such that the onset of the percolation transition is exactly at the thermal critical point.

Figure 1 shows the threshold value of the percolation cumulant Π in the 2D Ising model as a function of the bond probability p_B. For $p_B \approx p_{\mathrm{CK}}$ we plotted the value of Π for the largest lattice we took (1000^2), as Π changes sensibly with the lattice size for the reason we explained above. Very near the critical probability p_{CK}, the infinite volume limit behaviour cannot be reproduced even on the largest lattice we have used. Therefore, the value of Π at the threshold on the largest lattice still differs from the asymptotic universal value by some amount and the corresponding point in plots like figure 1 does not sit on the relative plateau.

The vertical dashed line in the plot marks the minimal probability $p_{\mathrm{CK}} = 1 - \exp(-2J/kT_c) = 0.585\,78$. We see that the cumulant is quite stable to the right and to

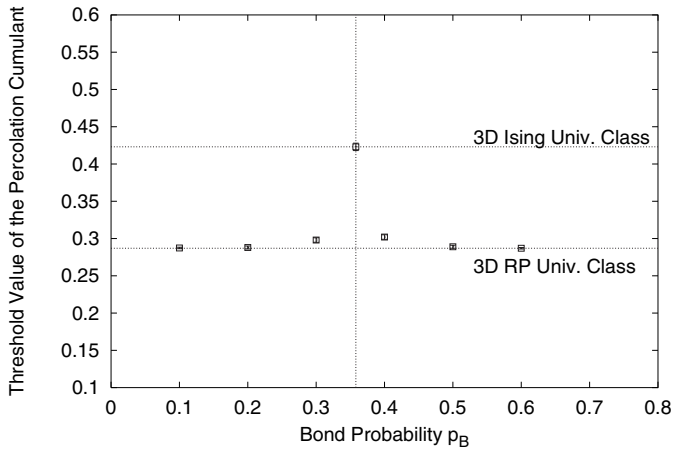

Figure 2. 3D Ising model: variation of Π at the percolation temperature T_p with the bond weight p_B.

the left of p_{CK}, and that the plateau values correspond to two different universality classes. For $p_B < p_{\mathrm{CK}}$ the site-bond clusters percolate at a temperature $T_p < T_c$, and the percolation exponents are in the random percolation universality class. This is to be expected, because if the percolation transition occurs at $T_p \neq T_c$, the physical correlation length $\xi(T_p)$ is finite; therefore, clusters separated by a distance $d > \xi(T_p)$ are not influenced by each other, and the long-range behaviour is the same as in random percolation. When $p_B > p_{\mathrm{CK}}$, $T_p = T_c$ and the exponents belong to a special universality class (we call it 2D $Z(2)$ site percolation universality class because it includes the pure site percolation case, $p_B = 1$, and is the same for all $Z(2)$ models we studied). For $p_B = p_{\mathrm{CK}}$ we recover the Fortuin–Kasteleyn mapping [3] and the site-bond clusters are the exact critical droplets of the system. We see that in this case the Π threshold value does not lie on either of the plateaux, since the exponents are now in a different universality class, i.e. in the class of the 2D Ising model.

In three dimensions the situation looks quite different. We performed simulations of the 3D Ising model, using several values for the bond probability above and below $p_{\mathrm{CK}} = 1 - \exp(-2J/kT_c) = 0.358\,08\ldots$. We studied the variation of Π with p_B as we did in two dimensions: the result is shown in figure 2. For $p_B \neq p_{\mathrm{CK}}$, the percolation temperature T_p is different from the magnetization temperature T_c and we recover the 3D random percolation exponents. Only for $p_B = p_{\mathrm{CK}}$ is $T_p = T_c$ and we could eventually get the thermal critical indices.

We note that, even if the 3D pattern is very different from the 2D one, the special (Fortuin–Kasteleyn) probability p_{CK} is still the smallest probability for which the site-bond clusters can form a percolating cluster at T_c (for $p_B < p_{\mathrm{CK}}$, $T_p < T_c$ and all clusters at T_c are finite).

We want to verify whether this is also valid for other 3D spin systems, and we analyse here the $O(2)$, or XY, model. For $O(n)$ models a rigorous mapping between percolation and critical behaviour was established in [7]. The droplets are built in two steps:

(i) choose a random vector $\mathbf{r}$ of $O(n)$;
(ii) bind together any pair of nearest-neighbouring spins $\mathbf{s_i}$, $\mathbf{s_j}$ with the probability

$$p(i,j) = 1 - \exp\{\min[0, -2\beta(\mathbf{s_i}\cdot\mathbf{r})(\mathbf{s_j}\cdot\mathbf{r})]\} \qquad (\beta = J/kT). \tag{5}$$

Such droplets are just the clusters devised by Wolff in his famous algorithm [15] for $O(n)$ spin systems. We see that only pairs of spin vectors having both a positive/negative projection

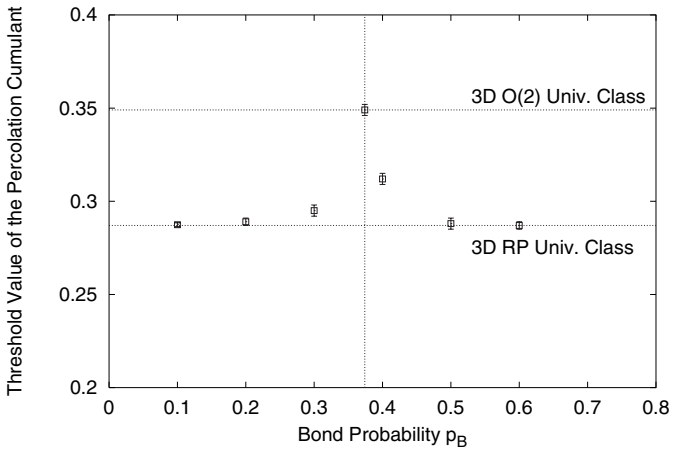

Figure 3. 3D $O(2)$ model: variation of Π at the percolation temperature T_p with the bond weight p_B.

on the random vector **r** can be joined to each other. The random vector **r**, therefore, divides the spin space into two hemispheres, separating the spins which have a positive projection onto it from the ones which have a negative projection. The droplets are made out of spins which all lie either in one or the other hemisphere. In this respect, we can again speak of 'up' and 'down' spins, as for the Ising model. In addition to that, the bond probability is local, since it explicitly depends on the spin vectors $\mathbf{s_i}$ and $\mathbf{s_j}$, and not only on the temperature like the Fortuin–Kasteleyn factor.

The situation is similar to the 2D continuous Ising model we considered in [9], and we proceeded in the same way, i.e. we reduced the $O(2)$ configurations to Ising configurations, according to the sign of the projection of the spins on **r**, so disregarding the length of the projection. The bond weight p_B we introduced is the same for each pair of nearest-neighbouring sites.

For our simulations we applied the Wolff algorithm and used four lattices: 24^3, 48^3, 72^3 and 96^3. At each run, 40 000 to 100 000 measurements were taken. We tuned the bond probability so as to make the percolation point coincide with the magnetization point. At the end we found the same scenario that we had seen for the Ising model (figure 3).

Our estimate of the minimal bond probability is $p_{\mathrm{CK}} = 0.374(1)$. Finally we calculated the critical indices of the percolation transition when $p_B = p_{\mathrm{CK}}$. Figure 4 shows the corresponding finite-size scaling plots of the percolation strenght P (top) and the average cluster size S (bottom) at the critical point. The χ^2 of the fits improves considerably if the smallest lattice (24^3) is excluded, and this is why we put only three points in the plots.

The critical indices we extracted agree with the thermal $O(2)$ values (table 1).

3.3. SU(2) pure gauge theory

The deconfining transition from hadronic matter to a plasma of quarks and gluons has been the object of intensive investigations over the last two decades. Though the concrete goal is to try to produce a quark–gluon plasma by means of high energy heavy ion collisions in the laboratory, i.e. in tiny and in general non-equilibrated fireballs, it is crucial from a theoretical point of view to study the ideal situation of an infinite system of strongly interacting matter in thermal equilibrium at a temperature T. This could be effectively done after the discovery of

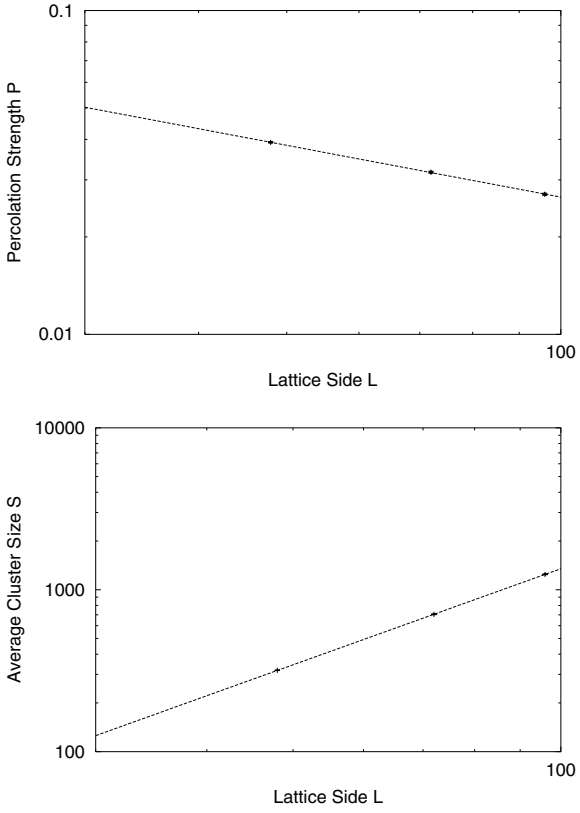

Figure 4. 3D $O(2)$ model: log–log plots of P (top) and S (bottom) at T_c versus the lattice side L for the minimal site-bond clusters.

Table 1. Critical percolation indices for the site-bond clusters when $p_B = p_{\rm CK}$; for comparison we also report the $O(2)$ thermal exponents (from [16]).

	β_p/ν_p	γ_p/ν_p	D
3D $O(2)$	0.5189(3)	1.9619(5)	2.4808(8)
Percolation exponents	0.530(15)	1.971(13)	2.484(7)

the lattice approach [17], and indeed finite temperature quantum chromodynamics (QCD) has been extensively simulated on the lattice since then.

The group that rules the gauge invariance of QCD is $SU(3)$, which is non-Abelian. Because of that the gauge fields are self-interacting and it makes sense to study systems constituted only by gluons. This simpler situation is described by the so-called $SU(3)$ pure gauge theory. Since any $SU(N)$ group is non-Abelian, the study of the relative pure gauge theories may be of interest also for $N \neq 3$.

Suppose we have a d-dimensional box containing gluons at a temperature T. The discretization of spacetime returns a $(d + 1)$-dimensional lattice, with N_σ spacings in each space direction and N_τ spacings in the imaginary time (or temperature) direction. The partition function of finite temperature $SU(N)$ pure gauge theories on this lattice takes the form

$$\mathcal{Z}(N_\sigma, N_\tau; g^2) = \int \prod_{links} \mathrm{d}U_{ij} \exp[-S(U)] \tag{6}$$

where $S(U)$ is the Wilson action

$$S(U) = \frac{2N}{g^2} \sum_{plaq} \left(1 - \frac{1}{N} \mathrm{Re\ Tr}\, UUUU\right). \tag{7}$$

Here g is the (temperature-dependent) coupling and U_{ij} the so-called link variable, which is a function of the gauge fields set between a pair of nearest-neighbouring sites i and j. The product in equation (6) runs over all links of the lattice, the sum in equation (7) over all the smallest closed paths (plaquettes), which are formed by four links; $UUUU$ is the product of the link variables corresponding to each side of a plaquette.

All $SU(N)$ pure gauge theories undergo a transition from a phase in which the gluons are bound in glueballs to a phase of free gluons. Such deconfining transition is due to the spontaneous breaking of a global $Z(N)$ symmetry which results from the periodicity of the gauge fields in the temperature direction. We recall that $Z(N)$ is the *centre* of the $SU(N)$ group, i.e. the subset of those elements which commute with each element of $SU(N)$. Such elements are the matrices obtained by multiplying the unit matrix by the complex N-roots of 1.

The order parameter is the lattice average of the Polyakov loop, defined as

$$L = \left\langle \left| \frac{1}{N_\sigma^3} \sum_{\vec{x}} L_{\vec{x}} \right| \right\rangle \tag{8}$$

with

$$L_{\vec{x}} = \frac{1}{N} \mathrm{Tr} \prod_{t=1}^{N_\tau} U_{\vec{x};t,t+1} \tag{9}$$

The product in (9) runs over all the U in the temperature direction taken at a given spatial site $\vec{x}$. We remark that the absolute value in equation (8) is necessary because otherwise, on a finite lattice, one would always get $L = 0$ due to the $Z(N)$ symmetry. In the confined phase $L = 0$, whereas at deconfinement $L \neq 0$. The main features of the deconfining transition are then all in the Polyakov loop configurations one obtains by projecting out the temperature direction of the lattice through the matrix product of equation (9).

There are conjectures suggesting that the deconfining transition of $SU(N)$ pure gauge theories is intimately related to the magnetization transition of the N-state Potts model, with which they share the $Z(N)$ symmetry [18]. In particular, it was predicted that if the deconfining transition is second order, the critical indices are in the universality class of the corresponding Potts model: this prediction has been confirmed by computer simulations without exceptions. This is actually the reason why we investigated simple Potts-like spin systems. As our droplet prescription seems to work for these models, we tried to see whether it is correct for $SU(N)$ gauge theories as well, at least in the cases in which the deconfining transition is continuous. In this way we would have for the confinement–deconfinement transition the same geometrical picture as for magnetization in the Potts model. If we take a typical Polyakov loop configuration of an $SU(N)$ theory at a certain temperature, one can identify domains where the average of $L_{\vec{x}}$ is close to one of the complex N-roots of one. Such domains can be seen as local regions of deconfinement. As long as these islands are finite, deconfinement remains a local phenomenon and the whole system is in the confined phase. When one of these islands percolates, i.e. it becomes infinite, we can talk of deconfinement as a global phase of the system.

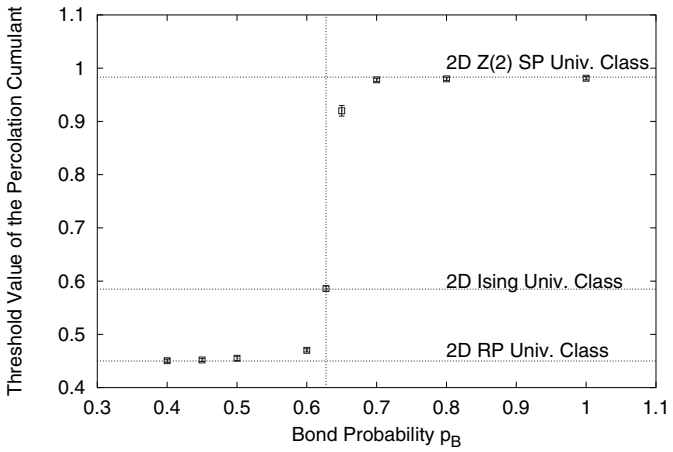

Figure 5. $2+1\,SU(2)$: variation of Π at the percolation temperature T_p with the bond weight p_B.

Table 2. Critical percolation indices for the site-bond clusters of $2+1\,SU(2)$ ($N_\tau = 2$) when $p_B = p_{\rm CK}$, compared with the values of the 2D Ising droplets.

	β_p/ν_p	γ_p/ν_p	D	Π at T_p
2D Ising	$1/8 = 0.125$	$7/4 = 1.75$	$15/8 = 1.875$	0.585(1)
$SU(2)$ percolation	0.140(19)	1.761(17)	1.882(18)	0.586(5)

We examined the simple case of $SU(2)$ pure gauge theory, in two and three space dimensions. In both cases we focused on one and the same lattice regularization, corresponding to $N_\tau = 2$; however, the result should hold for any N_τ (see the conclusions). We stress that for $SU(2)$ the Polyakov loop $L_{\vec{x}}$ is a real number, which can take all possible values in a range (from -1 to $+1$ according to our normalization choice). We are in the same situation as in the continuous Ising model, and again we will consider only the sign of $L_{\vec{x}}$, disregarding its absolute value. So, the Polyakov loop configurations reduce themselves to Ising-like configurations, that we treated in the same way as in the previous section.

The results for $2+1\,SU(2)$ have already been established in [9]: we disclose them here for completeness. We carried out our simulations on four different lattices, $64^2 \times 2$, $96^2 \times 2$, $128^2 \times 2$ and $200^2 \times 2$. For each run we collected from 10 000 to 40 000 measurements. The result of our analysis is identical to that for the $Z(2)$ spin systems (figure 5).

If we compare figure 5 to figure 1 we see no differences, except in the value of the minimal bond probability (for $SU(2)$, $p_{\rm CK} = 0.6275(5)$).

This is truly remarkable and shows the substantial analogy of the two cases. The exponents we calculated for $p_B = p_{\rm CK}$ are in good accord with the 2D Ising ones, which shows that the minimal clusters are critical droplets for $SU(2)$ as well (table 2).

Let us now turn to $3+1\,SU(2)$. We used four lattices, namely $12^3 \times 2$, $20^3 \times 2$, $30^3 \times 2$ and $40^3 \times 2$. The number of measurements ranged from 20 000 to 50 000 for each run. The pattern we derived for Π at the percolation threshold (figure 6) is the same as that for the 3D Ising model (see figure 2), only the value of $p_{\rm CK}$ is different (here $p_{\rm CK} = 0.385(4)$).

We note in particular that for $p_B = p_{\rm CK}$ the threshold value of Π sits at the height corresponding to the 3D Ising universality class. The exponents relative to the transition

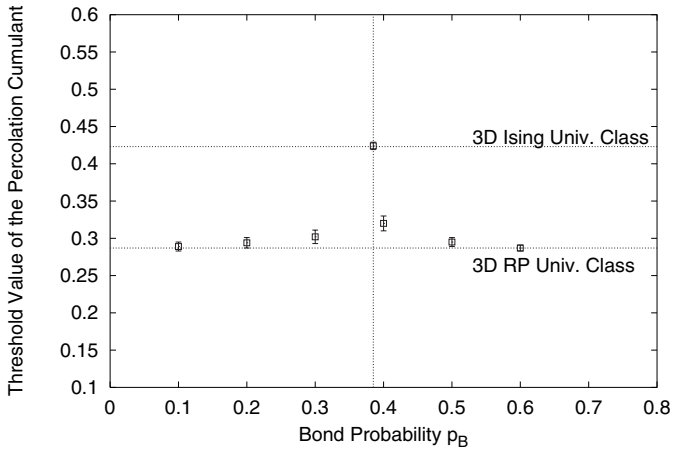

Figure 6. $3+1$ $SU(2)$: variation of Π at the percolation temperature T_p with the bond weight p_B.

Table 3. Critical percolation indices for the site-bond clusters of $3+1$ $SU(2)$ ($N_\tau = 2$) when $p_B = p_{\rm CK}$, compared with the values of the 3D Ising droplets (from [19]).

	β_p/ν_p	γ_p/ν_p	D	Π at T_p
3D Ising	0.5187(14)	1.963(7)	2.4817(21)	0.423(5)
$SU(2)$ percolation	0.54(3)	1.962(8)	2.477(9)	0.424(5)

of the minimal clusters agree within errors with the 3D Ising exponents, as we expected (table 3).

4. Conclusions

We have seen that the phase transitions of many systems can be interpreted as a geometrical percolation transition of simple site-bond clusters, as in the Ising model. We have found a general criterion to define the 'correct' bond probability $p_{\rm CK}$ of geometrical connection between nearest-neighbouring sites carrying spins of the same sign: $p_{\rm CK}$ is the minimal probability for which it is still possible to have a percolating cluster at the critical temperature T_c of the system. We remark that this criterion looks a bit artificial, since it imposes by hand the coincidence of the percolation with the thermal threshold. So, if one studies a new model, it would be impossible to define the droplets until one finds the critical temperature of the system. However, this is not relevant for us, as our aim was just to show that the droplets exist. The results presented here show that the criterion of the minimal probability, which was proved to be successful on bidimensional lattices [9], holds also at higher dimensions. To the extent of the models we investigated, our recipe provides indeed a solution of the problem, valid even for systems for which 'exact' droplets could not so far be identified (models with competitive interactions, $SU(2)$). The generality of the solution is clearly shown by its validity for a complex theory such as $SU(2)$: the effective theory of the Polyakov loop for $SU(N)$ gauge theory consists of a mixture of many different interactions, short- and long-ranged, ferromagnetic and antiferromagnetic, including couplings between more than two spins (like plaquette-interactions, six-spin couplings, etc) and self-interactions. Our analysis is entirely numerical, but the result is most likely exact, as it is for Ising. Some of the first works on

correlated percolation [4, 20], based on renormalization group analyses, revealed the existence of a limited number of fixed points for the possible percolation models built in q-state Potts models. That means that all percolation systems that one can define on the configurations of q-state Potts models can be divided into a small number of classes, each of them characterized by a well-defined set of critical indices: the droplet prescription is then most likely not unique.

The criterion we propose puts in evidence the key role of geometrical connectivity in the mechanism of the phase transition. This is easy to understand for the models we have considered, where the nearest-neighbour coupling is by far the most important compared to eventual longer ranged interactions and determines the critical behaviour. However this may not be valid when the strength of other couplings is comparable to the nearest-neighbour one: in this case the phase transition could be influenced as well by the other interactions (e.g., the critical indices might change), and a droplet definition based only on nearest-neighbour connections is probably inadequate.

In our analysis of the models with $Z(2)$ symmetry, we have seen that the essential spin feature for the droplet definition is the $Z(2)$ variable, i.e. the sign of the spin. In fact, we built the clusters in the same way, no matter whether the spin is discrete as in Ising or continuous as in $SU(2)$. This shows once more the crucial role played by the $Z(2)$ symmetry, whose spontaneous breaking is indeed responsible for the phase transition. Since the relationship between $SU(N)$ pure gauge theories and $Z(N)$ spin models holds for any N, our result should be valid for all $SU(N)$ pure gauge theories with a continuous deconfining transition, i.e. also for $SU(3)$ and $SU(4)$ in $2+1$ dimensions. The special lattice regularization of the gauge theory does not play a role, as the critical behaviour has the same features for any N_τ.[1] So, we found that the confinement–deconfinement phase transition of $SU(N)$ pure gauge theory, if second order, is equivalent to a percolation transition of special site-bond clusters of like-signed Polyakov loops: this is the most important result of our work. We remark, however, that the droplets we defined for $SU(N)$ gauge theories 'live' in the Polyakov loop configurations that one obtains after projecting out the imaginary time direction of the initial $(d+1)$-dimensional lattice; therefore, our clusters cannot help to devise a cluster algorithm for $SU(N)$ gauge theories. Recent studies show that percolation can also help to describe the chiral transition of special fermion lattice models [21].

We focused only on systems with a continuous second-order transition because the percolation phenomenon is typically smooth. This does not mean that we cannot use percolation to describe discontinuous phase changes. The Fortuin–Kasteleyn mapping [3] is valid for any q-state Potts model in any dimension, and the equivalence between the magnetization and the percolation order parameter holds for first-order phase transitions as well. In this case at the critical point there is coexistence of a paramagnetic and a ferromagnetic phase, which correspond to two different 'geometrical phases', characterized by small and large clusters, respectively. Therefore, the percolation order parameter jumps at the threshold exactly as the magnetization (from zero to a non-zero value). Because of that we believe that the criterion of the minimal bond probability we adopted here can be directly extended to systems undergoing a discontinuous phase transition. Now it is, however, more difficult to support the idea of droplets because there are no critical indices to reproduce. One could nevertheless try to see whether the percolation variables (P, S) vary near T_c like their thermal counterparts (m, χ). In this way one could study in particular the 'physical' confinement–deconfinement transition of $SU(3)$ pure gauge theory in $3+1$ dimensions (quenched QCD).

[1] We have numerical evidence that our result holds as well for $N_\tau = 4$ in $2+1$ $SU(2)$; we do not present it here because the statistics is not high.

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